Preferences as binary relations

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Chapter 1

Definition

- A subset of ordered pairs of a set X is called a binary relation.
- Formally, R is a binary relation on X if $R \subseteq X \times X$.
- Usually we write $x \; R \; y$ if $(x, y) \in R$

Relation as Directed Graphs

Let be R a relation on a set A. A direct graph representation of relation R is $G = (A, E)$ where A is the set of nodes and E the set of direct edges where

$$(a, b) \in R \Longleftrightarrow (a, b) \in E(\text{an arrow from } a \text{ to } b)$$

Relation as Matrices

Let be R a relation on a set A. The matrix representation of relation R is $M_R = [m_{ab}]_{(a,b)\in R}$ where

$$\begin{cases} m_{ab} = 1 & \text{if } (a, b) \in R \\ m_{ab} = 0 & \text{if } (a, b) \notin R \end{cases}$$

Properties

A binary relation R is

- Reflexive if for every $x \in X$, $x\ R\ x$;
- Irreflexive if for every $x \in X$, $\mathrm{not}(x\ R\ x)$
- Complete if for every $x, y \in X$, $x\ R\ y$ or $y\ R\ x$ (possibly both);
- Weakly complete if for every $x, y \in X$, $x \neq y \Longrightarrow [x\ R\ y$ or $y\ R\ x]$ (possibly both);
- Symmetric if for every $x, y \in X$, $[x\ R\ y \Longrightarrow y\ R\ x]$;
- Asymmetric if for every $x, y \in X$, $[x\ R\ y \Longrightarrow \mathrm{not}(y\ R\ x)]$;
- Antisymmetric if for every $x, y \in X$, $[x\ R\ y \text{ and } y\ R\ x \Longrightarrow x = y]$;
- Transitive if for every $x, y, z \in X$, $[x\ R\ y \text{ and } y\ R\ z \Longrightarrow x\ R\ z]$;
- Negatively transitive if for every $x, y, z \in X$, $[\mathrm{not}(x\ R\ y) \text{ and } \mathrm{not}(y\ R\ z) \Longrightarrow \mathrm{not}(x\ R\ z)]$;
- Semi-transitive if for every $x, y, z, t \in X$, $[(x\ R\ y) \text{ and } (y\ R\ z)] \Longrightarrow [(x\ R\ t) \text{ or } (t\ R\ z)]$
- Ferrers if for every $x, y, z, t \in X$, $[(x\ R\ y) \text{ and } (z\ R\ t)] \Longrightarrow [(x\ R\ t) \text{ or } (z\ R\ y)]$.

Relations P and I from R

For a binary relation R on X, we define a symmetric part I and an asymmetric part P as follows: for all $x, y \in X$

- $x \ I \ y$ if $[x \ R \ y$ and $y \ R \ x]$
- $x \ P \ y$ if $[x \ R \ y$ and $\mathrm{not}(y \ R \ x)]$

Concatenation of two binary relations

Let be $\mathcal{R}$ and $\mathcal{R}'$ two binary relations on X. For all $x, y \in X$

$$x \; \mathcal{R} \bullet \mathcal{R}' \; y \iff \text{there exists } z \in X \text{ s.t. } [x \; \mathcal{R} \; z \text{ and } z \; \mathcal{R}' \; y]$$

Proposition

Let be $\mathcal{R}$ a binary relation on X.

1. $\mathcal{R}$ *transitive* $\implies \mathcal{R} \bullet \mathcal{R} \subseteq \mathcal{R}$ (*i.e.* $\mathcal{R}^2 \subseteq \mathcal{R}$)
2. $\mathcal{R}$ *asymmetric* $\implies \mathcal{R}$ *irreflexive*
3. $\mathcal{R}$ *complete* $\iff \mathcal{R}$ *reflexive and weakly complete*
4. $\mathcal{R}$ *asymmetric and negative transitive* $\implies \mathcal{R}$ *transitive*
5. $\mathcal{R}$ *complete and transitive* $\implies \mathcal{R}$ *negative transitive*

Definition

- A binary relation R on X that is reflexive, symmetric and transitive is called an equivalence relation.
- A binary relation R on X is a preorder if R is reflexive and transitive.
- A binary relation R on X is a weak order or a complete preorder if R is complete and transitive.
- A binary relation R on X is a total order or a linear order if R is complete, antisymmetric and transitive.

Exercise 1

Let be $\mathcal{B}$ a binary relation on a set $X = \{a, b, c, d, e, f\}$ defined by:

$$a \ \mathcal{B} \ a, \ a \ \mathcal{B} \ b, \ a \ \mathcal{B} \ c, \ a \ \mathcal{B} \ d, \ a \ \mathcal{B} \ e, \ a \ \mathcal{B} \ f$$

$$b \ \mathcal{B} \ b, \ b \ \mathcal{B} \ c, \ b \ \mathcal{B} \ d, \ b \ \mathcal{B} \ e, \ b \ \mathcal{B} \ f$$

$$c \ \mathcal{B} \ c, \ c \ \mathcal{B} \ d, \ c \ \mathcal{B} \ e, \ c \ \mathcal{B} \ f$$

$$d \ \mathcal{B} \ b, \ d \ \mathcal{B} \ c, \ d \ \mathcal{B} \ d, \ d \ \mathcal{B} \ e$$

$$e \ \mathcal{B} \ d, \ e \ \mathcal{B} \ e, \ e \ \mathcal{B} \ f$$

$$f \ \mathcal{B} \ e, \ f \ \mathcal{B} \ f$$

1. Give a matrix and a graphical representation of $\mathcal{B}$
2. Is $\mathcal{B}$ reflexive? symmetric? asymmetric? transitive? negative transitive? semi-transitive?

Exercise 2

Let us consider a binary relation $\mathcal{B}$ on $A = \{a; b; c; d; e; f\}$ defined as follows:

$a\ \mathcal{B}\ a;\ a\ \mathcal{B}\ b;\ a\ \mathcal{B}\ c;\ a\ \mathcal{B}\ d;\ a\ \mathcal{B}\ e;\ a\ \mathcal{B}\ f;$
$b\ \mathcal{B}\ a;\ b\ \mathcal{B}\ b;\ b\ \mathcal{B}\ c;\ b\ \mathcal{B}\ d;\ b\ \mathcal{B}\ e; b\ \mathcal{B}\ f;$
$c\ \mathcal{B}\ b;\ c\ \mathcal{B}\ c;\ c\ \mathcal{B}\ d;\ c\ \mathcal{B}\ e;\ c\ \mathcal{B}\ f;$
$d\ \mathcal{B}\ c;\ d\ \mathcal{B}\ d;\ d\ \mathcal{B}\ e;\ d\ \mathcal{B}\ f;$
$e\ \mathcal{B}\ c;\ e\ \mathcal{B}\ d;\ e\ \mathcal{B}\ e;\ e\ \mathcal{B}\ f$
$f\ \mathcal{B}\ e;\ f\ \mathcal{B}\ f$

- $\mathcal{R}$ is said to be a *semi-order* if it is complete, semi-transitive and de Ferrers.
- The binary relation $\mathcal{R}^+$ on A defined by for all $a, b \in A$:

$$a\ \mathcal{R}^+\ b \iff \text{ for all } c \in A, \begin{cases} b\ \mathcal{R}\ c \Rightarrow a\ \mathcal{R}\ c \\ \text{and} \\ c\ \mathcal{R}\ a \Rightarrow c\ \mathcal{R}\ b \end{cases}$$

  is called the *trace* of $\mathcal{R}$.

1. Give the *trace* of the relation $\mathcal{B}$ above.
2. Is $\mathcal{B}$ a semi-order?

How to extend a partial pre-order to a complete preorder?

By applying a topological sorting when there is no strict cycle in the preferences.

Idea of the numerical representation

We try to construct a binary relation $\succsim$ on X such that there exists a numerical function $f : X \longrightarrow \mathbb{R}$ satisfying the property:

$$x \succsim y \Longleftrightarrow f(x) \geq f(y)$$

In general, $\succsim$ is assumed to be a preorder.

- $x \succsim y$ means x is at least as good as y
- $\succ$ is the asymmetric part of $\succsim$
- $\sim$ is the symmetric part of $\succsim$

Proposition

Let be $\succsim$ a preorder (complete) on X representable by a function $f : X \longrightarrow \mathbb{R}$ i.e. $\forall x, y \in X, x \succsim y \iff f(x) \geq f(y)$

The following two properties are equivalent:

(i) $v : X \longrightarrow \mathbb{R}$ *is a function representing* $\succsim$

(ii) *There exists a strictly increasing function* $\varphi : f(X) \longrightarrow \mathbb{R}$ *such that* $v = \varphi \circ f$

Remark

f is an ordinal scale (See Chapter 2).

Separability

A binary relation $\succsim$ on X is said to be separable if there exists a countable set $Z \subseteq X$ such that, for every $x, y \in X \subseteq Z$,

$$\text{if } x \succ y \text{ then there exists } z \in Z \text{ such that } x \succsim z \succsim y$$

Theorem (For every X)

The following are equivalent

(i) $\succsim$ *is complete, transitive, and separable*

(ii) *There is a function* $u : X \longrightarrow \mathbb{R}$ *that representing* $\succsim$

Reference

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